A harmonic oscillator perturbed by the potential $\lambda x^2/(1+gx^2)$

1981 J. Phys. A: Math. Gen. 14 377

(http://iopscience.iop.org/0305-4470/14/2/013)

J. Phys. A: Math. Gen. **14** (1981) 377–378. Printed in Great Britain

A harmonic oscillator perturbed by the potential $\lambda x^2/(1+gx^2)$

K V Bhagwat

Nuclear Physics Division, Bhabha Atomic Research Centre, Trombay, Bombay 400085, India

Received 16 July 1980

Abstract. The eigenvalues of the Hamiltonian $H = -d^2/dx^2 + x^2 + \lambda x^2/(1+gx^2)$, with λ and g as parameters, are studied. We present a simple numerical procedure where the input data are known exactly and the requirements on the computer memory are not stringent. The results agree very well with the available calculations reported earlier.

The solution of the eigenvalue problem

$$\boldsymbol{H}\psi = E\psi, \tag{1}$$

where $\boldsymbol{H} = -d^2/dx^2 + x^2 + \lambda x^2/(1+gx^2)$, has recently attracted attention due to its applications in a variety of problems (see Mitra (1978 and references therein), Kaushal (1979)). Mitra (1978) obtained the first three eigenvalues of (1) numerically. Choosing as the basis the well known solutions of

$$\boldsymbol{H}_0\phi_n = E_n\phi_n, \tag{2}$$

with $\boldsymbol{H}_0 = -d^2/dx^2 + x^2$, he constructed the explicit matrix representation of $\boldsymbol{H}$. By actually diagonalising a finite-dimensional matrix obtained by truncating the infinite-dimensional matrix $\boldsymbol{H}$, Mitra was able to obtain the first three eigenenergies for a set of values of the parameters λ and g. His calculation becomes involved on two counts. Firstly, he had to evaluate the matrix elements of $(1+gx^2)^{-1}$, which can best be done by using a recurrence relation. However, none of the non-zero elements can be evaluated in a closed form. Secondly, in his case, one needs to store the entire matrix in the computer memory. This puts a severe restriction on the dimensionality of the 'approximate' representation of $\boldsymbol{H}$.

We observe that with the same basis $\{\phi_n\}$ the matrix corresponding to $(1+gx^2)\,H$ has a simple structure. Non-zero elements occur only along three diagonals and these matrix elements are known *exactly*. Therefore, instead of solving (1) one may solve

$$(1+gx^2)\boldsymbol{H}\psi = E(1+gx^2)\psi. \tag{3}$$

It may be noticed that $(1+gx^2)$, being positive definite, is invertible and hence equation (1) is equivalent to equation (3). Further, equation (3) (like equation (1)) remains invariant under parity operation, hence odd and even parity solutions form invariant subspaces. These can be treated independently. In each (even or odd) subspace, the relevant matrices become tridiagonal. The eigenvalue problem reduces to finding the zeros of a tridiagonal determinant. The roots can be accurately determined, the process

0305-4470/81/020377+02$01.50

being facilitated by the fact that the principal minors of $\det\{(1+gx^2)\boldsymbol{H}-E(1+gx^2)\}$ form a Sturm sequence (see, for example, Hammarling (1970)). The following inequality, which can be easily established, provides the required lower and upper bounds on the nth energy level:

$$\boldsymbol{H}_0 < \boldsymbol{H} < \boldsymbol{H}_0 + \boldsymbol{H}_0/(1+g\boldsymbol{H}_0). \tag{4}$$

Starting with even parity solutions of equation (2), we generate the appropriate 10×10 matrices and then compute the first two roots correct to five decimal places. The dimensionality is increased by ten and the calculation repeated until an agreement to five decimal places is obtained between successive computations. The same procedure is repeated for the odd parity solutions. We thus obtain the first four eigenenergies. The advantage of the present method is that all the input data are known exactly. Secondly, many matrix elements are identically zero, and the procedure for finding the roots does not require the zero elements to be stored in the computer memory. Thus, for an approximate calculation with $N\times N$ matrices, the number of elements to be stored is only $(2N+1)$. The results are given in table 1. The first three levels can be compared with the results of Mitra. The agreement is very good.

Table 1. The numerical results for the first four energy levels of a harmonic oscillator perturbed by the potential $\lambda x^2/(1+gx^2)$ are presented for typical values of the parameters λ and g.

g \ λ	0·1	1·0	10	100
0·1	1·043 174	1·380 533	3·250 264	9·976 199
	3·120 089	4·079 900	9·619 087	29·781 266
	5·181 112	6·667 938	15·729 379	49·292 816
	7·231 014	9·166 578	21·591 056	68·513 244
1·0	1·024 112	1·232 353	2·782 331	9·359 432
	3·051 498	3·507 397	7·417 534	26·706 007
	5·058 980	5·689 803	10·701 033	41·441 139
	7·064 899	7·648 212	13·388 354	53·839 119
10	1·005 948	1·059 298	1·580 028	5·793 965
	3·008 817	3·088 091	3·879 039	11·572 215
	5·008 291	5·082 864	5·832 771	13·628 777
	7·009 050	7·090 384	7·903 174	15·988 477
100	1·000 855	1·008 465	1·084 138	1·836 461
	3·000 989	3·009 840	3·098 330	3·983 112
	5·000 936	5·009 317	5·092 807	5·928 395
	7·000 999	7·009 849	7·098 468	7·984 464

The author is grateful to Dr R Subramanian for useful discussions and comments.

References

Hammarling S J 1970 *Latent Roots and Latent Vectors* (London: Adam Hilger) pp 62–70
Kaushal R S 1979 *J. Phys. A: Math. Gen.* **12** L253–8
Mitra A K 1978 *J. Math. Phys.* **19** 2018–22